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Notes on Fourier Series: Strong Approximation

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Two results are proved about strong approximation of Fourier series. The first, which goes in the positive direction, is the best possible refinement of an inequality of L. Leindler. The second is related to the inverse problem and provides a unified treatment to many earlier results.

1. INTRODUCTION

Let f be an everywhere continuous 2π-periodic function, $s_k(x)$ the kth partial sum of its Fourier series and $E_n = E_n(f)$ the error in the best uniform approximation of f by trigonometric polynomials of order at most n.

Since the 1963 work of G. Alexits and D. Králik [1], many authors have investigated the so-called strong approximation of Fourier series. Among their results, perhaps the most important one is the inequality

$$\frac{1}{n}\sum_{k=n+1}^{2n} |s_k - f|^p \leqslant K_p E_n^p(f) \qquad (p>0; n=1, 2,...) \tag{1.1}$$

due to L. Leindler [3] which can be used to estimate strong means of the type

$$\left\{\sum_{k=1}^{\infty} t_{nk}\, |s_k(x) - f(x)|^p\right\}^{1/p} \qquad (p>0,\ t_{nk} \geqslant 0)$$

(see [3, 5]).

A natural generalization is the following: instead of x^p $(x \geqslant c)$, let us consider a non-negative monotone increasing continuous function $\phi(x)$ $(x \geqslant 0, \phi(0)=0)$. For such a ϕ, we proved in [7]

THEOREM A. *The means*

$$\frac{1}{n+1}\sum_{k=0}^{n} \phi(|s_k(x) - f(x)|)$$

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tend to zero for every continuous f if and only if there exists a constant A such that

$$\phi(\tau)\leqslant e^{A\tau} \qquad (\tau\in(0,\infty)). \tag{1.2}$$

An analog of (1.1) is

THEOREM 1. *The inequality*

$$\frac{1}{n}\sum_{k=n+1}^{2n}\phi(|s_k(x)-f(x)|)\leqslant K\phi(E_n) \qquad (x\in[0,2\pi],\ n=1,2,\ldots) \tag{1.3}$$

holds for every 2π-periodic continuous function f if and only if there is a constant $A>0$ such that (1.2) *holds true and*

$$\phi(2\tau)\leqslant A\phi(\tau) \qquad (\tau\in(0,1)). \tag{1.4}$$

Theorem 1 has many consequences of which we mention only one:

COROLLARY. *If* (1.2) *and* (1.4), *then, for every $\beta>0$ and for $f^{(r)}\in \operatorname{Lip}\alpha$* $(0<\alpha\leqslant 1)$,

$$\phi^{-1}\left(\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}\phi(|s_k-f|)\right)$$
$$\leqslant\phi^{-1}\left(\frac{K}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}\phi((k+1)^{-r-\alpha})\right).$$

This generalizes a number of previous results, and the last inequality cannot, in general, be improved.

Let us now turn to the inverse problem. We ask what smoothness properties of f follow from the relation

$$\sup_{0\leqslant x\leqslant\pi}\sum_{k=0}^{\infty}\phi(|s_k(x)-f(x)|)<\infty? \tag{1.5}$$

Here, again, $\phi(x)$ is supposed to be monotone increasing and continuous for $x\geqslant 0$, with $\phi(0)=0$. For example, G. Freud [2] proved that if $\phi(x)=x^p$ $(p>1)$, then (1.5) implies $f\in\operatorname{Lip}(1/p)$. Generalizing this result, we proved [6]

THEOREM B. *If ϕ is convex or concave, then* (1.5) *implies*

$$\omega(f;\delta)\leqslant K\delta\int_\delta^{2\pi}\frac{\phi^{-1}(x)}{x^2}\,dx \qquad (\delta\in[0,\pi]) \tag{1.6}$$

and this is the best possible result, since, for a function f satisfying (1.5),

$$\omega(f;\delta) \geqslant c\delta \int_{\delta}^{2\pi} \frac{\phi^{-1}(x)}{x^2}\,dx \qquad (c>0;\ \delta\in[0,\pi]).$$

($\omega(f;\delta)$ *is the modulus of continuity of* f.)

If $\phi(x)=x^p$ $(0<p<1)$, then Theorem B gives only $f\in \text{Lip } 1$ although much more is true (see [4]):

THEOREM C. *If* $f(x)=x^p$ $(0<p<1)$, *then* (1.5) *implies*

$$E_n(f)\leqslant Kn^{-1/p}.$$

Theorem C yields, among other things, that f is $[v/p]$ times continuously differentiable (v, $0<v<1$, is arbitrary).

Now we show that Theorem C and Freud's result are valid for very general ϕ's.

THEOREM 2. *If* ϕ *satisfies*

$$\phi(a\tau)\leqslant \tfrac{1}{2}\phi(\tau) \qquad (\tau\in[0,1]) \tag{1.7}$$

with a constant $0<a<1$, *then* (1.5) *implies*

$$E_n(f)\leqslant K\phi^{-1}\left(\frac{1}{n}\right). \tag{1.8}$$

As a corollary we get that under the only assumption (1.7), we have (1.6).

2. PROOF OF THEOREM 1

I. *Sufficiency.* We shall use the inequality (see [5])

$$\frac{1}{r}\sum_{i=1}^{r} |s_{k_i}(x)-f(x)| \leqslant A_1 E_n(f)\log\frac{2n}{r} \tag{2.1}$$

where $1\leqslant r\leqslant n$, the indices $n<k_1<k_2<\cdots<k_r\leqslant 2n$ are arbitrary, and A_1 is an absolute constant. This gives at once that if $\lambda>0$, $x\in(0,2\pi]$ and $\mu_{n,\lambda}(x)$ denotes the number of those $n<k\leqslant 2n$ for which $|s_k(x)-f(x)|>\lambda E_n$, then

$$\mu_{n,\lambda}(x)\leqslant 2n\exp(-\lambda/A_1).$$

Let $y \in (0, (2AA_1)^{-1})$. By (1.4), $\phi(y) \geqslant y^{A_2}$, A_2 a constant. Thus, by (1.2),

$$\sum_{k=2/y}^{\infty} \phi(yk)\, e^{-k/A_1} \leqslant \sum_{k=2/y}^{\infty} \exp(k(Ay - A_1^{-1}))$$

$$\leqslant \sum_{k=2/y}^{\infty} e^{-k/(2A_1)} \leqslant Ke^{-1/(yA_1)} \leqslant K\phi(y). \qquad (2.2)$$

(K denotes constants, not necessarily the same.)

From (1.4) we obtain

$$\sum_{k=1}^{2/y} \phi(yk)\, e^{-k/A_1} \leqslant \sum_{k=1}^{\infty} \phi(y)\, k^{A_3} e^{-k/A_1} \leqslant K\phi(y)$$

with some A_3 and this, together with (2.2), gives

$$\sum_{k=1}^{\infty} \phi(yk)\, e^{-k/A_1} \leqslant K\phi(y) \qquad (y \in (0, (2AA_1)^{-1})).$$

Now the proof of (1.3) is easy:

$$\frac{1}{n} \sum_{k=n+1}^{2n} \phi(|s_k(x) - f(x)|)$$

$$= \frac{1}{n} \sum_{k=1}^{\infty} \sum_{E_n(k-1) \leqslant |s_l(x) - f(x)| \leqslant E_n k} \phi(|s_l(x) - f(x)|)$$

$$\leqslant \frac{1}{n} \sum_{k=1}^{\infty} \phi(kE_n)\, \mu_{n,k-1}(x) \leqslant K \frac{1}{n} \sum_{k-1}^{\infty} 2n\phi(kE_n)\, e^{-k/A_1} \leqslant K\phi(E_n)$$

provided n is so large that $E_n = E_n(f) \in (0, (2AA_1)^{-1})$.

II. *Necessity.* The necessity of (1.2) follows from Theorem A.

To prove (1.4) let us assume on the contrary that there is a sequence $\{a_n\}$ with $a_n \downarrow 0$ $(n \to \infty)$ and $\phi(40a_n) > n\phi(20a_n)$. We may also suppose $\sum_{k=n+1}^{\infty} a_k < a_n$ $(n = 1, 2, \ldots)$. For $m < n$ let

$$Q_{n,m}(x) = \sum_{i=1}^{m} \left(\frac{\cos(n-i)\, x}{i} - \frac{\cos(n+i)\, x}{i} \right) \equiv 2 \sin nx \sum_{i=1}^{m} \frac{\sin ix}{i}$$

be the well-known Fejér polynomial and let

$$f(x) = \sum_{k=1}^{\infty} a_k Q_{2^{k+2}, 2^k}(x).$$

As $|Q_{n,m}| \leqslant 10$ we get

$$E_{2^{n+2}}(f) \leqslant E_{2^{n+2}-2^n-1}(f) \leqslant 10 \sum_{k=n}^{\infty} a_k \leqslant 20a_n \qquad (n=1, 2, \ldots)$$

and so

$$\begin{aligned}
&\frac{1}{2^{n+2}} \sum_{k=2^{n+2}}^{2 \cdot 2^{n+2}} \phi(|s_k(0)-f(0)|) \\
&\geqslant \frac{1}{2^{n+2}} \sum_{k=2^{n+2}+1}^{2^{n+2}+2^n} \phi\left(a_n \sum_{i=k-2^{n+2}}^{2^n} \frac{1}{i}\right) \\
&\geqslant \frac{1}{2^{n+2}} \sum_{k=2^{n+2}+1}^{2^{n+2}+2^n \cdot e^{-10}} \phi(a_n \log(2^n/(k-2^{n+2}))) \\
&\geqslant \frac{1}{2^{n+3}} \frac{2^n}{e^{40}} \phi(40a_n) \geqslant cn\phi(20a_n) \geqslant cn\phi(E_{2^{n+2}}),
\end{aligned}$$

i.e., (1.3) is not satisfied, which proves the necessity of (1.4).

3. Proof of Theorem 2

First we prove the following

LEMMA. *If $f \in C_{2\pi}$ and $0 < c < \frac{1}{2}$, then, for every natural number n, either*

(i) $E_{2n}(f) \leqslant 2cE_n(f)$ *or*

(ii) *there exists a point $x_n \in (0; 2\pi]$ such that the number of the indices k for which*

$$n < k \leqslant 2n, \qquad |s_k(f; x_n) - f(x_n)| > cE_n(f)$$

is at least $A(c/|\log c|)\, n$ where A denotes an absolute constant.

Proof. Let

$$H_n(x) = \{k \mid n < k \leqslant 2n, |s_k(f; x) - f(x)| > cE_n\}.$$

If $2cE_n < E_{2n}$, then, using (2.1), we obtain for an $x_n \in (0; 2\pi]$,

$$\begin{aligned}
2cE_n < E_{2n} &\leqslant \left\| \frac{1}{n} \sum_{k=n+1}^{2n} |s_k - f| \right\|_c = \frac{1}{n} \sum_{k=n+1}^{2n} |s_k(x_n) - f(x_n)| \\
&= \frac{1}{n} \sum_{k \notin H_n(x_n)} + \frac{1}{n} \sum_{k \in H_n(x_n)} \leqslant cE_n + A_1 \frac{|H_n(x_n)|}{n} E_n \log \frac{4n}{|H_n(x_n)|},
\end{aligned}$$

i.e.,

$$c < A_1 \frac{|H_n(x_n)|}{n} \log \frac{4n}{|H_n(x_n)|}$$

which gives

$$|H_n(x_n)| > A \frac{c}{|\log c|} n$$

and this is exactly (ii).

Let us turn to the proof of Theorem 2. Let $c = a/2$ in the above Lemma. If, for a given n, we have $E_{2n} > aE_n$, then Lemma (ii) and (1.5) give

$$A \frac{a/2}{|\log(a/2)|} n\phi\left(\frac{a}{2} E_n\right) \leqslant K$$

by which

$$E_{2n} \leqslant E_n \leqslant C\phi^{-1}\left(\frac{1}{2n}\right)$$

for some C. On the other hand, if $E_{2n} \leqslant aE_n$, then assuming that $E_n \leqslant C\phi^{-1}(1/n)$ is already proved, we obtain, from (1.7),

$$E_{2n} \leqslant aE_n \leqslant aC\phi^{-1}\left(\frac{1}{n}\right) \leqslant C\phi^{-1}\left(\frac{1}{2n}\right).$$

Thus an easy induction gives $E_{2^n}(f) \leqslant C\phi^{-1}(1/2^n)$ which is equivalent to (1.8) (take into account that, by (1.7), we have

$$\phi^{-1}(x) \leqslant a^{-1}\phi^{-1}\left(\frac{x}{2}\right).$$

We have proved our theorem.

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